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Finite Element Analysis of a Scattering Problem

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Abstract. A finite element method for the solution of a scattering problem for the reduced wave equation is formulated and analyzed. The method involves a reformulation of the problem on a bounded domain with a nonlocal boundary condition. The space of trial functions includes piecewise polynomial functions and functions arising from spherical harmonics.

1. Introduction. In this paper we develop a finite element method for the numerical solution of scattering problems for the reduced wave equation. In [4] and [9] there are reported various schemes for solving electromagnetic scattering problems. [6] and [18] give an engineering discussion of the finite element treatment of the radiation boundary conditions. [7] and [10] deal with mathematical aspects of the analysis of coupled finite element-boundary solution procedures.

Here we present a new numerical approximation technique with the following features:

(i) The subspaces are a combination of finite element subspaces inside the absorbing domain and spherical harmonics in the exterior domain. In addition to introducing a novel way of coupling the spherical harmonics with the field inside the body, the method has the feature that no finite elements are required outside the absorbing body; see, e.g., [9].

(ii) The stiffness matrices are nonsingular and Gaussian elimination without pivoting can be used for the solution of the linear system.

The formulation and the analysis of the present method as applied to the reduced Maxwell equations shall be discussed in a forthcoming paper. The problem considered here is important in many applications. Our work was motivated by a study of the biological effects of microwave radiation. For this problem a computer program, FEMS, has been written which uses the method described in this paper. A discussion of the program and some numerical results will be presented elsewhere.

In Section 2 we formulate our problem and in Section 3 we give the variational formulation and prove a series of lemmas in order to obtain our main result, Theorem 3.1. In Section 4 we describe the finite element procedure for our problem. We also give in this section an error estimate and a brief discussion of the finite-dimensional subspaces involved.

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2. Formulation of the Problem. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with smooth boundary Γ and $\Omega_0 = \mathbf{R}^3 \setminus \bar{\Omega}$ be the exterior domain. Let (u, v) and $\|u\|$ denote, respectively, the inner product and norm in $L_2(\Omega)$. Similarly, let $\langle u, v \rangle$ and $|u|$ denote the inner product and norm in $L_2(\Gamma)$. We shall use the Sobolev spaces $H^s(\Omega)$, $H^s(\Gamma)$, and we let $\|u\|_s$ and $|u|_s$, respectively, designate the norms in these spaces. We let n denote the exterior normal to Ω and let u_n denote the normal derivatve of u. Suppose $\kappa(x)$ is a bounded, complex valued function on $\mathbf{R}^3$ such that

$$(2.1) \qquad \begin{cases} \operatorname{Im} \kappa(x)^2 \geqslant a > 0, & x \in \Omega, \\ \kappa(x) = \kappa_0 > 0, & x \in \Omega_0. \end{cases}$$

Let $f(x)$ be a function on $\mathbf{R}^3$ with $f(x) = 0$ for $x \in \Omega_0$, and let u_0 satisfy $\Delta u_0(x) + \kappa_0^2 u_0(x) = 0$, $x \in \mathbf{R}^3$. We shall be concerned with the Problem I: to find $u(x)$ in $H^1_{\text{loc}}(\mathbf{R}^3)$ satisfying, with $|x| = r$,

$$(2.2) \qquad \Delta u + \kappa(x)^2 u = f \quad \text{in } \mathbf{R}^3 \setminus \Gamma,$$

$$(2.3) \qquad \begin{cases} u - u_0 = O(r^{-1}), & r \to \infty, \\ \dfrac{\partial}{\partial r}(u - u_0) - i\kappa_0(u - u_0) = o(r^{-1}), & r \to \infty. \end{cases}$$

We shall also introduce an auxiliary problem which we call Problem $\mathrm{II}_\pm$. To this end, let $s \in \mathbf{R}^1$, and $g \in H^s(\Gamma)$ be given. We seek a function v such that

$$(2.4) \qquad \Delta v_\pm + \kappa_0^2 v_\pm = 0 \quad \text{in } \Omega_0,$$

$$(2.5) \qquad v_\pm(x) = g(x), \qquad x \in \Gamma,$$

$$(2.6) \qquad \begin{aligned} v_\pm(x) &= O(r^{-1}), & r \to \infty, \\ \partial v_\pm/\partial r \mp i\kappa_0 v_\pm &= o(r^{-1}), & r \to \infty. \end{aligned}$$

It is known [11] that the above problem has a unique solution, which can be expressed in terms of a Green's function $G_\pm(x, y)$ by

$$(2.7) \qquad v_\pm(x) = \int_\Gamma \frac{\partial G_\pm}{\partial n_y}(x, y) g(y)\, ds_y.$$

The function $G_\pm(x, y)$ is smooth for $x \in \Omega_0, y \in \Gamma$, but becomes singular as $x \to y$. It is also known that the normal derivative $\partial v_\pm/\partial n$ is well defined on Γ, and $\partial v_\pm/\partial n \in H^{s-1}(\Gamma)$. We let $K_\pm$: $H^s(\Gamma) \to H^{s-1}(\Gamma)$ be the mapping defined by $K_\pm\, g = \partial v_\pm/\partial n$. Then for each $s \in \mathbf{R}^1$, $K_\pm$ is a bounded map from $H^s(\Gamma)$ into $H^{s-1}(\Gamma)$. Also, $K_\pm$ is a pseudodifferential operator on $H^s(\Gamma)$ of order 1. If, in a neighborhood N of a point $x^* \in \Gamma$, the surface Γ coincides with the plane $x_3 = 0$, and, in N, Ω lies in the half space $x_3 < 0$, it may be shown, using the Fourier transformation, that the symbol of $K_\pm$ at x^* is

$$(2.8) \qquad \sigma_\pm(x^*, \xi_1, \xi_2) = -\left(\xi_1^2 + \xi_2^2\right)^{1/2}.$$

For the theory of elliptic boundary value problems with pseudodifferential operators, see, for example, [2].

3. Variational Formulations. To give a finite element procedure for the approximate solution of (2.2) and (2.3), we reformulate the problem by introducing a bilinear form. To this end let γ denote the trace operator, restricting a function to

Γ. Thus, γ: $H^1(\Omega) \to H^{1/2}(\Gamma)$ is a bounded operator. Let $\langle g, h \rangle$ denote the inner product in $L^2(\Gamma)$, and also the pairing between $g \in H^s(\Gamma)$ and $h \in H^{-s}(\Gamma)$. We define a bilinear form on $H^1(\Omega) \times H^1(\Omega)$ by

$$B(u, w) = -(\nabla u, \nabla w) + (\kappa^2 u, w) + \langle K_+ \gamma u, \gamma w \rangle.$$

Clearly B is a bounded form. Using B, we formulate Problem III: find $u \in H^1(\Omega)$ such that

$$B(u, w) = B(u_0, w) + (f - \Delta u_0 - \kappa^2 u_0, w), \qquad w \in H^1(\Omega). \tag{3.1}$$

Using the following lemma, we see that Problem III gives a reformulation of Problem I.

LEMMA 3.1. *Let u solve Problem* I, *and suppose $u \in H^2(\Omega)$. Then u solves* III.

Proof. Set $z = u - u_0$. From Green's formula,

$$(f - \Delta u_0 - \kappa^2 u_0, w) = (\Delta z + \kappa^2 z, w) = -(\nabla z, \nabla w) + (\kappa^2 z, w) + \langle z_n, w \rangle.$$

In Ω_0, z solves Problem II_+ with $g = \gamma z$. Hence $z_n = K_+ \gamma z$, and we have (3.1).

Let

$$H^2_\pm(\Omega) = \{u \in H^2(\Omega): u_n = K_\pm \gamma u\}.$$

Then $H^2_\pm(\Omega)$ is a closed subspace of $H^2(\Omega)$. We define mappings

$$A_\pm: H^2_\pm(\Omega) \to H^0(\Omega), \quad A_+ u = \Delta u + \kappa^2 u, \quad A_- u = \Delta u + \bar{\kappa}^2 u.$$

Using Green's formula we have

$$(A_+ u, w) = B(u, w), \qquad u \in H^2_+(\Omega), w \in H^1(\Omega). \tag{3.2}$$

We show

LEMMA 3.2.

$$(A_+ u, w) = (u, A_- w), \qquad u \in H^2_+(\Omega), w \in H^2_-(\Omega), \tag{3.3}$$

$$a\|u\| \leqslant \|A_\pm u\|, \qquad u \in H^2_\pm(\Omega). \tag{3.4}$$

Proof. Let $u \in H^2_+(\Omega)$, $w \in H^2_\pm(\Omega)$, so

$$(A_+ u, w) = -(\nabla u, \nabla w) + (\kappa^2 u, w) + \langle u_n, w \rangle. \tag{3.5}$$

Let U denote the solution of Problem II_+ with $g = \gamma u$, and let W denote the solution of Problem $\mathrm{II}_\pm$ with $g = \gamma w$.

Since $u_n = K_+ u$, we have $U_n = u_n$ on Γ. Let B_R denote the ball with center 0 and radius R, and with surface S_R. Choose R so that $B_R \supset \Omega$, and let $\Omega_{0,R} = B_R \cap \Omega_0$. Then, since $\Delta u + \kappa_0^2 u = 0$ in Ω_0,

$$\langle u_n, w \rangle = \int_{\Omega_{0,R}} \left[-\nabla U \cdot \overline{\nabla W} + \kappa_0^2 U \overline{W}\right] dx + \int_{S_R} \overline{W} \frac{\partial u}{\partial r}\, d\sigma.$$

Inserting this formula in (3.5) and using (2.6), we obtain

$$\begin{aligned}(A_+ u, w) &= -(\nabla u, \nabla w) + (\kappa^2 u, w) \\ &\quad + \lim_{R \to \infty} \left\{ \int \left[-\nabla U \cdot \nabla \overline{W} + \kappa_0^2 U \overline{W}\right] dx + i\kappa_0 \int_{S_R} U \overline{W}\, dS \right\}.\end{aligned} \tag{3.6}$$

Similarly, if $w \in H^2_-(\Omega)$ and $u \in H^2_\pm(\Omega)$, we obtain

$$(3.7)\qquad \begin{aligned}(A_-w, u) &= -(\nabla w, \nabla u) + (\bar{\kappa}^2 w, u)\\ &+ \lim_{R\to\infty}\left\{\int\left[-\nabla W\cdot\nabla\bar{U} + \kappa_0^2 W\bar{U}\right]dx - i\kappa_0\int_{S_R} W\bar{U}\,dS\right\}.\end{aligned}$$

Comparing (3.6) and (3.7), we get (3.3). Setting $u = w$ in (3.6) or (3.7), taking the imaginary part, and using (2.1), we obtain (3.4).

Remark. Using (3.2), (3.4), and a limiting argument, we obtain

$$(3.8)\qquad a\|u\|^2 \leqslant |B(u, u)|, \qquad u \in H^1(\Omega).$$

The density of $H^2_\pm(\Omega)$ in $H^1(\Omega)$ follows from Lemma 3.4 below and density properties of interpolation spaces.

We regard $A_\pm$ as an unbounded operator on $L^2(\Omega)$ with domain $H^2_\pm(\Omega)$. With this stipulation we have

LEMMA 3.3. *$A_\pm$ is a closed, densely defined, invertible operator on $L^2(\Omega)$ and $(A_\pm)^* = A_\mp$.*

Proof. Since $H^2_\pm(\Omega)$ contains smooth functions which vanish near Γ, $H^2_\pm(\Omega)$ is dense in $L^2(\Omega)$. From (3.4) it follows that $A_\pm$ is (1-1). We show that the boundary condition $u_n = K_\pm u$ covers the operator $A_\pm$ in Ω. For suppose that, in a neighborhood N of a point $x^* \in \Gamma$, the surface Γ coincides with the plane $x_3 = 0$, and, in N, Ω lies in the half space $x_3 < 0$. The covering condition at x^* requires that if $z(t)$ satisfies the equations

$$\begin{gathered} z'' - (\xi_1^2 + \xi_2^2)z = 0, \qquad t < 0,\\ z(t) \to 0, \qquad t \to -\infty,\\ z'(0) = \sigma_\pm(x^*, \xi_1, \xi_2)z(0),\end{gathered}$$

then $z(t) \equiv 0$. Using (2.8), we easily verify this, so the covering condition is satisfied at x^*. Since the covering condition is preserved under a change of independent variables, it follows that the covering condition holds at each point of Γ. Hence the a priori inequality holds [2, p. 101], and, using (3.4), we obtain

$$(3.9)\qquad \|u\|_2 \leqslant c\|A_\pm u\|, \qquad u \in H^2_\pm(\Omega).$$

From (3.9) it easily follows that $A_\pm$ is a closed operator and that the range of $A_\pm$ is a closed subspace of $L_2(\Omega)$. Since Problem I has a solution for all smooth f, the range of $A_\pm$ is dense in $L_2(\Omega)$. Hence the range of $A_\pm$ is $L_2(\Omega)$, and $(A_\pm)^{-1}$ is a bounded operator on $L_2(\Omega)$. Setting $A_+u = f$, $A_-w = g$ in (3.3), we obtain

$$(f, A_-^{-1}g) = (A_+^{-1}f, g), \qquad f, g \in L_2(\Omega).$$

Hence $(A_\pm^{-1})^* = A_\mp^{-1}$, so, from [17, Chapter 8, Theorem 6.2], $(A_\pm)^* = A_\mp$.

LEMMA 3.4. *$[H^2_\pm(\Omega), L_2(\Omega)]_{1/2} = H^1(\Omega)$.*

Proof. Since $H^2_\pm(\Omega) \subset H^2(\Omega)$, we see that the inclusion $\subset$ holds. We must show the converse inclusion. We make use of the spaces $H^{r,s}$ defined in [8]. Given $f \in H^1(\Omega)$ we must show that there is a function $u(x, y) \in H^{2,1}(\Omega, R^1)$ such that

$$(3.9)\qquad u(x, 0) = f(x), \qquad x \in \Omega,$$

$$(3.10)\qquad u_n(x, y) = K_\pm u(x, y), \qquad x \in \Gamma, y \in R^1.$$

To this end we must first discuss the operator $K_\pm$ on the space $H^{r,s}(\Gamma, R^1)$. Let $-\Lambda$ be the Laplace-Beltrami operator on Γ; then $I + \Lambda$ is positive definite and

$$(I + \Lambda)^{\mp 1/2}: H^r(\Gamma) \to H^{r\pm 1}(\Gamma).$$

Therefore $(I + \Lambda)^{-1/2}K_\pm$ is a bounded operator on $H^r(\Gamma)$ and may be defined as a bounded operator on $H^r(\Gamma \times R^1)$. Since Λ is a second order differential operator, by [8, Lemma 4.2],

$$\Lambda: H^{r,s}(\Gamma, R^1) \to H^{r-2,s-2s/r}(\Gamma, R^1)$$

is a bounded operator. Hence, by interpolation,

$$(I + \Lambda)^{1/2}: H^{r,s}(\Gamma, R^1) \to H^{r-1,s-s/r}(\Gamma, R^1)$$

is a bounded operator. Writing $K_\pm = (I + \Lambda)^{1/2} \cdot (I + \Lambda)^{-1/2}K_\pm$,

$$K_\pm: H^{r,s}(\Gamma, R^1) \to H^{r-1,s-s/r}(\Gamma, R^1)$$

is a bounded operator.

Returning to the solution of (3.9), (3.10) we set $f_0(x)w = f(x)$. Since $\gamma f \in H^{1/2}(\Gamma)$, by [8, Lemma 4.1], there is a $g_0(x, y) \in H^{3/2,3/4}(\Gamma, R^1)$ such that $g_0(x, 0) = f(x)$, $x \in \Gamma$. Then we find a $u(x, y) \in H^{2,1}(\Omega, R^1)$ such that

$$\begin{aligned} u(x, 0) &= f_0(x), && x \in \Omega,\\ u(x, y) &= g_0(x, y), && x \in \Gamma, y \in R^1,\\ u_n(x, y) &= g_1(x, y), && x \in \Gamma, y \in R^1. \end{aligned}$$

Hence u satisfies (3.9), (3.10) and the proof is complete.

We now establish the "inf sup condition" for the form B.

THEOREM 3.1. *There is a $c > 0$ such that for each $u \in H^1(\Omega)$ there is a $v \in H^1(\Omega)$ such that $|B(u, v)| \geqslant c\|u\|_1\|v\|_1$.*

Proof. Regarding $A_\pm$ as a bounded invertible map of $H^2_\mp(\Omega) \to L^2(\Omega)$, we extend A_+ to a map of $L^2(\Omega) \to H^2_-(\Omega)'$ as follows. If $f \in L_2(\Omega)$, we consider $A_+ f$ as a linear functional on $H^2_-(\Omega)$ according to the formula

$$(A_+ f)(w) = (f, A_- w), \qquad w \in H^2_-(\Omega).$$

In particular, if $f \in H^2_+(\Omega)$, we see that

$$(A_+ f)(w) = (A_+ f, w),$$

so this definition agrees with the previous definition of A_+. The extended map is also easily seen to be (1-1) and invertible. By interpolation we find that

$$A_+: \left[H^2_+(\Omega), L_2(\Omega)\right]_{1/2} \to \left[L^2(\Omega), H^2_-(\Omega)'\right]_{1/2}$$

is a bounded, (1-1) invertible map. From Lemma 3.4 and the dual properties of interpolation [5] it follows that $A_+: H^1(\Omega) \to H^1(\Omega)'$ is a bounded, (1-1) invertible map. By taking the limit in (3.2), we find that for $u \in H^1(\Omega)$, $w \in H^1(\Omega)$,

$$(A_+ u)(w) = B(u, w), \qquad u, w \in H^1(\Omega).$$

Now let $w \in H^1(\Omega)$ be given. Select $\xi \in H^1(\Omega)'$ such that

$$\xi(w) = \|w\|_1, \quad \|\xi\|_{H^1(\Omega)'} = 1. \tag{3.11}$$

Let $u = A_+^{-1}\xi \in H^1(\Omega)$. Then from (3.11) we have

$$B(u, w) = \|w\|_1 \geqslant c\|w\|_1\|u\|_1,$$

where $c = \|A_+^{-1}\|^{-1}$, and where the norm refers to the map A_+^{-1}: $H^1(\Omega)' \to H^1(\Omega)$.

For our estimate we also require

LEMMA 3.5. *There are constants* $c_i > 0$ $(i = 1, 2)$ *such that for* $u \in H^1(\Omega)$

$$|B(u, u)| \geqslant c_1\|u\|_1^2 - c_2\|u\|_0^2.$$

Proof. We have

$$\begin{aligned} |B(u, u)| &\geqslant -\mathrm{Re}\, B(u, u) = (\nabla u, \nabla u) - \mathrm{Re}(\kappa^2 u, u) - \mathrm{Re}\langle K_+\gamma u, \gamma u\rangle \\ &\geqslant c_3\|u\|_1^2 - c_4\|u\|_0^2 - \mathrm{Re}\langle K_+\gamma u, \gamma u\rangle. \end{aligned} \tag{3.12}$$

Let Ω_1 be the region between Ω and $|x| < R$, where $R >$ diameter of Ω, and denote by U the extension of u to Ω_{ext}. Then

$$\int_{\Omega_1} \left[-|\nabla U|^2 + \kappa_0^2|U|^2\right] dx = -\int_{|x|=R} \bar{U}\frac{\partial U}{\partial r}\, ds + \langle K_+\gamma u, \gamma u\rangle.$$

Hence

$$\mathrm{Re}\langle K_+\gamma u, \gamma u\rangle \leqslant \kappa_0^2\int_{\Omega_1} |U|^2\, dx + \int_{|x|=R} |\bar{U}U_r|\, ds. \tag{3.13}$$

Now we select R so that κ_0^2 is not an eigenvalue of the problem

$$\Delta z + \lambda z = 0 \quad \text{in } \Omega_1, \qquad z(x) = 0, \quad x \in \partial\Omega_1.$$

Then we may consider U as the solution of the well-posed Dirichlet problem

$$\Delta U + \kappa_0^2 U = 0 \quad \text{in } \Omega_1, \qquad U \text{ given on } \partial\Omega_1. \tag{3.14}$$

We now show that, for any real s, there is a constant $c(s) > 0$ such that

$$\|U\|_{H^s(\Omega_1)} \leqslant c(s)\|U\|_{H^{s-1/2}(\partial\Omega_1)}. \tag{3.15}$$

(See, e.g., [3, Theorem 2.4.2] for a related assertion.) To prove this, we first note that if ϕ satisfies

$$\Delta\phi + \kappa_0^2\phi = f \quad \text{in } \Omega_1, \qquad \phi = g \quad \text{on } \partial\Omega_1,$$

then

$$\|\phi\|_{H^{s+2}(\Omega_1)} \leqslant c_1(s)\left[\|f\|_{H^s(\Omega_1)} + \|g\|_{H^{s+3/2}(\Omega_1)}\right], \qquad s \geqslant 0. \tag{3.16}$$

Setting $\phi = U$, we obtain (3.15) for $s \geqslant 2$. Next, we choose ϕ so that $g = 0$. Then, by Green's formula,

$$\int_{\Omega_1} Uf\, dx = -\int_{\partial\Omega_1} U\frac{\partial\phi}{\partial n}\, ds.$$

Hence, using (3.16) and the trace inequality

$$\left\|\frac{\partial\phi}{\partial n}\right\|_{H^{s+1/2}(\partial\Omega_1)} \leqslant c_2(s)\|\phi\|_{H^{s+2}(\Omega_1)}, \qquad s \geqslant 0,$$

we obtain

$$\left|\int_{\Omega_1} Uf\, dx\right| \leqslant c_3(s)\|U\|_{H^{-s-1/2}(\partial\Omega_1)}\|f\|_{H^s(\Omega_1)}, \qquad s \geqslant 0.$$

Dividing both sides by the norm of f and taking the supremum over smooth f, we obtain (3.15) for $s \leqslant 0$. Now let A be the solution operator for the problem (3.14). Thus, we have shown that

$$A: H^{s-1/2}(\partial\Omega_1) \to H^s(\Omega_1) \tag{3.17}$$

is a bounded operator for $s \leqslant 0$ and $s \geqslant 2$. By interpolation we conclude that (3.17) is a bounded operator for all s, so (3.15) holds for all s.

Using (3.15) with $s = \frac{1}{2}$, and (3.13), we obtain

$$\operatorname{Re} \langle K_+ \gamma u, \gamma u \rangle \leqslant c_5 \left\{ |u|_0^2 + \int_{|x|=R} \left(|U|^2 + |U_r|^2 \right) ds \right\}. \tag{3.18}$$

We now use (2.7) and the fact that S_R is at a positive distance from Γ, so $\partial G_\pm(x, y)/\partial n_y$ is a smooth function for $x \in S_R, y \in \Gamma$. We obtain

$$\int_{|x|=R} \left(|U|^2 + |U_r|^2 \right) ds \leqslant c_6 |u|_0^2.$$

Using these in (3.18),

$$\operatorname{Re}\langle K_+ \gamma u, \gamma u \rangle \leqslant c_7 |u|_0^2 \leqslant c_7 |u|_{\theta-1/2}^2 \leqslant c_8 \|u\|_\theta^2, \qquad \tfrac{1}{2} < \theta < 1.$$

From (3.12) and the inequality $\|u\|_\theta^2 \leqslant \varepsilon \|u\|_1^2 + c(\varepsilon)\|u\|_0^2$, we obtain the result.

4. The Discrete Problem. To formulate our discrete approximation to Problem I, we specify a finite-dimensional subspace $S \subset H^1(\Omega)$, and, in analogy with (3.1), we seek a $\tilde{u} \in S$ such that

$$B(\tilde{u}, w) = B(u_0, w) + \left(f - \Delta u_0 - \kappa^2 u_0, w \right), \qquad w \in S. \tag{4.1}$$

We shall refer to $\tilde{u}$ as the approximate solution of Problem I, using the subspace S. We first show that the approximate solution $\tilde{u}$ is well defined.

LEMMA 4.1. *There is exactly one $\tilde{u} \in S$ satisfying* (4.1).

Proof. The equation (4.1) when written in terms of a basis for S, comprise a finite system of linear equations. To show that the system is nonsingular, it suffices to show that if $z \in S$ and if $B(z, w) = 0$ for all $w \in S$, then $z = 0$. Choosing $w = z$ and applying (3.8), we obtain

$$a\|z\|^2 \leqslant |B(z, z)| = 0.$$

To analyze the discretization error $u - \tilde{u}$, we shall show that B satisfies a discrete form of the inf sup condition. For this we first prove a weak form of the inf sup condition that may also be of use in other problems. (See Schatz [14] for a similar result.)

LEMMA 4.2. *Let H_i, $i = 0, 1, 2$, be three Hilbert spaces. Suppose $H_0 \supset H_1$ with compact injection. Let B be a bounded bilinear form on $H_1 \times H_2$ which satisfies: if $u \in H_1$ and $B(u, v) = 0$, $v \in H_2$, then $u = 0$. For $n = 1, 2, \ldots$, let $M_{in} \subset H_i$, $i = 1, 2$, be two finite-dimensional subspaces of equal dimension. Suppose $M_{2n} \subset M_{2n+1}$ and $\bigcup_n M_{2n}$ is dense in H_2. Suppose B satisfies the "weak* inf sup *condition": there are $c_i > 0$, $i = 1, 2$, such that for $u \in M_{1n}$ there is a $v \in M_{2n}$ such that*

$$|B(u, v)| \geqslant \left[c_1 \|u\|_{H_1} - c_2 \|u\|_{H_0} \right] \|v\|_{H_2}. \tag{4.2}$$

Then there is an integer $N > 0$ and a constant $c_3 > 0$ such that for $n \geqslant N$, if $u \in M_{1n}$, there is a $v \in M_{2n}$ such that

$$|B(u, v)| \geqslant c_3\|u\|_{H_1}\|v\|_{H_2}.$$

Proof. Let M'_{2n} denote the space of linear functionals on M_{2n} and define a map $L: M_{1n} \to M'_{2n}$ by

$$(Lu)(v) = \overline{B(u, v)}, \qquad v \in M_{2n}.$$

Then there is an integer $N > 0$ and a constant $c_4 > 0$ such that if $n \geqslant N$

$$\|u\|_{H_0} \leqslant c_4\|Lu\|_{M'_{2n}}, \qquad u \in M_{1n}. \tag{4.3}$$

For, if (4.3) does not hold, there are sequences $n_j \to \infty$ and $u_j \in M_{1n_j}$ such that

$$\|u_j\|_{H^0} = 1, \quad \|Lu_j\|_{M'_{2n_j}} \to 0.$$

Let $v_j \in M_{2n_j}$ be chosen to satisfy (4.2) and normalized so $\|v_j\|_{H_2} = 1$. Then from (4.2) it follows that

$$\|Lu_j\|_{M'_{2n_j}} \geqslant |(Lu_j)(v_j)| = |B(u_j, v_j)| \geqslant c_1\|u_j\|_{H_1} - c_2\|u_j\|_{H_0}.$$

Hence

$$c_1\|u_j\|_{H_1} \leqslant c_2 + O(1),$$

so u_j is a bounded sequence in H_1. Hence, selecting a subsequence, we may assume that $u_j \rightharpoonup u$ in H_1, $u_j \to u_0$ in H_0. Let $v \in H_2$ be arbitrary and, from density, let $\tilde{v}_j \in M_{2,j}$ be chosen so that $\|v - \tilde{v}_j\|_{H_2} \to 0$. Then

$$\begin{aligned} |B(u_j, v)| &\leqslant |B(u_j, \tilde{v}_j)| + |B(u_j, v - \tilde{v}_j)| \\ &\leqslant \|Lu_j\|_{M'_{2n}}\|\tilde{v}_j\|_{H_2} + c\|u_j\|_{H_1}\|v - \tilde{v}_j\|_{H_2} \to 0. \end{aligned}$$

Hence $B(u, v) = 0$ for all $v \in H_2$. By our hypothesis, $u = 0$ which is a contradiction and proves (4.3). Now, for $u \in M_{1n}$, choose $v \in M_{2n}$ to satisfy (4.2) and with $\|v\|_{2n} = 1$. Then

$$\|Lu\|_{M'_{2n}} \geqslant |B(u, v)| \geqslant c_1\|u\|_{H^1} - c_2\|u\|_{H_0},$$

so we obtain from (4.3)

$$\|u\|_{H_1} \leqslant c_5\|Lu\|_{M'_{2n}}, \qquad n \geqslant N. \tag{4.4}$$

Now let $n \geqslant N$, $u \in M_{1n}$, and let $v \in M_{2n}$ satisfy

$$(Lu)(v) = \|Lu\|_{M'_{2n}}, \quad \|v\|_{H_2} = 1.$$

Then, using (4.4), we have

$$|B(u, v)| = \|Lu\|_{M'_{2n}} \geqslant c_5^{-1}\|u\|_{H_1}\|v\|_{H_2},$$

and the proof is complete.

We now show that our approximate method gives, in a quasi-optimal sense, as good an approximation to the solution as can be expected from the subspace that is being used.

THEOREM 4.1. *Let $S_j \subset H^1(\Omega)$ be an increasing family of finite-dimensional subspaces of $H^1(\Omega)$ such that $\bigcup S_j$ is dense in $H^1(\Omega)$. Let $u_j \in S_j$ be the approximate solution of Problem* I *using the subspace S_j. Then there is a constant $c > 0$,* *independent of j but depending on the family* $\{S_j\}$, *such that, if u is the solution of Problem* I,

$$\|u - u_j\|_1 \leqslant c \inf\{\|u - z\|_1 : z \in S_j\}. \tag{4.5}$$

Proof. In Lemma 4.2, we set $H_1 = H_2 = H^1(\Omega)$, $H_0 = H^0(\Omega)$. Then, using Lemma 3.5, we see that the hypotheses of Lemma 4.2 hold. Using Lemmas 4.1 and 4.2, we find that there is a $J > 0$ such that, for $j \geqslant J$, the hypotheses of [3, Theorem 6.2.1] hold. Hence there is a $c > 0$ such that, for $j \geqslant J$, (4.5) holds. Since the cases $j = 1, 2, \ldots, J - 1$ are finite in number, we see that (4.5) holds for all j, which proves the theorem.

To find the approximate solution $\tilde{u}$ using a subspace S, we select a basis $\{z_i\}$, $1 \leqslant i \leqslant m$, of S. Setting

$$A = [a_{ij}], \quad a_{ij} = B(z_i, z_j), \quad \underline{F} = [f_i],$$

$$f_i = B(u_0, z_i) + (f - \Delta u_0 - \kappa^2 u_0, z_i), \qquad 1 \leqslant i \leqslant m,$$

and writing $\tilde{u}(x) = \sum u_i z_i(x)$, $\underline{U} = [u_i]$, we see that (4.1) may be written as the matrix system $A\underline{U} = \underline{F}$. From Lemma 4.1, this matrix equation always has a solution. It is important to be able to handle large matrices. In finite element programs this is frequently done with sparse matrix routines. The aim of the next lemma is to show that the matrix A can be factored without pivoting, so the unknowns can be arranged to minimize the storage requirements of the matrix.

LEMMA 4.3. *We may write* $A = LU$, *where L and U are, respectively, left and right triangular.*

Proof. Let I be a subset of $\{1, \ldots, m\}$, and let A_I be the principal minor of A obtained by removing column j and row j for each $j \notin I$. Let $S_I \subset S$ denote the subspace spanned by $\{z_i, i \in I\}$. Then A_I is the matrix used in finding the approximate solution u_I using the subspace S_I. From Lemma 4.1, A_I is nonsingular. Hence [13], [15] the factorization $A = LU$ may be accomplished.

Our approximate method has a potential difficulty, in that the operator K_+, and hence the bilinear form B, is difficult to evaluate. We overcome this difficulty by a judicious choice of subspaces, which we now describe. Let $V_N \subset H^1_{\text{loc}}(\Omega_0)$ be a finite-dimensional collection of functions which satisfy

$$\Delta v + \kappa_0^2 v = 0 \quad \text{in } \Omega_0, \tag{4.6}$$

$$\begin{cases} v(x) = O(r^{-1}), & r \to \infty, \\ \dfrac{\partial v}{\partial r}(x) - i\kappa_0 v(x) = o(r^{-1}), & r \to \infty. \end{cases} \tag{4.7}$$

A specific choice of V_N arises, for example, from the separation of variables in spherical coordinates applied to (4.6). Suppose $0 \in \Omega$. Let $Y_{m,n}(\theta, \phi)$ be a surface harmonic, and let $h_n^1(p)$ be a spherical Bessel function [1, Chapter 10]. Then $v(r, \theta, \phi) = h_n^1(\kappa_0 r) Y_{m,n}(\theta, \phi)$ is a particular solution of (4.6), (4.7). We may take V_N to be the collection of all such solutions with $1 \leqslant n \leqslant N$.

Regarding the subspace V_N we shall make the following

Assumption 1. The set $\gamma(\bigcup_N V_N)$ is dense in $H^{1/2}(\Gamma)$. For the spherical harmonic subspaces described above, the density of $\gamma(\bigcup_N V_N)$ in $L_2(\Gamma)$ has been recently proved in [12]. See also [19].

We also require a collection of functions of finite element type. Let there be given a decompostion of $\mathbf{R}^3$ into simplices of maximum size h. Let W_h be the set of restrictions to Ω of continuous piecewise linear functions on this triangulation. Let $W_{h0} = W_h \cap H_0^1(\Omega)$.

The subspace of functions used in our variational principle is formed by combining the spaces W_{h0} and V_N. We describe two ways in which this can be done. For the first way, we pick a smooth function ζ such that $\zeta \equiv 1$ near Γ and $\zeta \equiv 0$ near 0. If $v \in V_N$, then ζv is a smooth function in Ω, so the resulting restriction of ζv to Ω is in $H^1(\Omega)$. We also let ζv denote this restriction. We then set $S_{hN}^1 = W_{h0} + \zeta V_N$. We have

LEMMA 4.4. *If the subspaces V_N satisfy Assumption* 1, *then the collection of all the subspaces S_{hN}^1, $h > 0$, $N = 1, 2, \ldots,$ is dense in $H^1(\Omega)$.*

Proof. Supposing the contrary, we have, for some $z \in H^1(\Omega)$, $z \neq 0$,

$$\int\int_\Omega [\nabla z \cdot \nabla u + zu]\, dx = 0, \qquad u \in W_{h,0} + \zeta V_N. \tag{4.8}$$

Setting $u \in W_{h0}$ and using the fact that the union of these spaces is dense in $H_0^1(\Omega)$, we find that (4.8) holds for all $u \in H_0^1(\Omega)$. Hence $-\Delta z + z = 0$ in Ω, and z has a normal derivative z_n on Γ with $z_n \in H^{-1/2}(\Gamma)$. Set $u = \zeta v$, $v \in V_N$. Then from (4.8) we obtain

$$\int_\Gamma z_n v\, d\sigma = 0. \tag{4.9}$$

By Assumption 1, (4.9) holds for all $v \in H^{1/2}(\Gamma)$. Hence $z_n = 0$, so $z = 0$, which is a contradiction.

Using Lemma 4.4, we may apply Theorem 4.1 to obtain an error estimate when the subspace S_{hN}^1 is used. This subspace, however, has a certain disadvantage. The support of the functions in ζV_N depends on the support of ζ, and hence is independent of h. As a result, the number of nonzero matrix elements in the stiffness matrix is $O(h^{-3})$. To avoid this problem, we now give a second choice of subspace and a modified variational principle which commits a "variational crime".

For given h and N, let $P_{hN}: V_N \to \gamma(W_h)$ be a linear map such that $P_{hN}v - v$ is small on Γ. To be precise, we assume that $\|\gamma(P_{hN}v - v)\|_{H^{1/2}(\Gamma)}$ is small. For example, if the surface Γ were a polyhedron, and if the triangulation conformed with Γ, we could define P_{hN} by piecewise linear interpolation. We let S_{hN}^2 be the collection of all functions $v \in W_h$ such that $\gamma v \in P_{hN}(V_N)$. In particular, $W_{h0} \subset S_{hN}^2$. We let $Q_{hN}: \gamma(W_h) \to V_N$ be a map such that $P_{hN}Q_{hN} = I$. Thus, Q_{hN} is a right inverse of P_{hN}. We define a bilinear form $\tilde{B}$ on S_{hN}^2 by

$$\tilde{B}(u, w) = -(\nabla u, \nabla w) + (\kappa^2 u, w) + \langle K_+ \gamma Q_{hN} u, \gamma w\rangle, \qquad u, w \in S_{hN}^2. \tag{4.10}$$

In analogy with (3.1) we define an approximate method as follows. We seek a $\tilde{u} \in S_{hN}^2$ such that

$$\tilde{B}(\tilde{u}, w) = B(u_0, w) + (f - \Delta u_0 - \kappa^2 u_0, w), \qquad w \in S_{hN}^2. \tag{4.11}$$

We remark that the subspace S_{hN}^2 depends not only on the spaces W_h and V_N, but also on the choice of the approximation operator P_{hN}. The bilinear form $\tilde{B}$

depends not only on S^2_{hN}, but on the choice of the right inverse Q_{hN}. It is not evident that the system (4.11) has a solution $\tilde{u}$. If there is a solution, the following theorem gives an error estimate for it. To state the theorem, we need another assumption on the family of subspaces V_N and approximation operators P_{hN}.

Assumption 2. The set $\gamma(\bigcup_N P_{hN}V_N)$ is dense in $H^{1/2}(\Gamma)$.

THEOREM 4.2. *Suppose that the subspaces* V_N *and maps* P_{hN} *satisfy Assumption* 2. *Then there is a constant* $c > 0$, *which does not depend on* h *or* N, *such that if* u *satisfies* (3.1) *and* $\tilde{u}$ *satisfies* (4.11), *then*

$$\|u - \tilde{u}\|_1 \leqslant c \inf\{\|u - u^*\|_1 : u^* \in S^2_{hN}\} + c\|(I - P_{hN})Q_{hN}\tilde{u}\|_{H^{1/2}(\Gamma)}. \tag{4.12}$$

Proof. Using Assumption 2, the proof of Lemma 4.4 shows that $\bigcup S^2_{hN}$ is dense in $H^1(\Omega)$. From Lemma 4.2, B satisfies the inf sup condition on S^2_{hN}. Let $u^* \in S^2_{hN}$ be arbitrary. Then there is a $w \in S^2_{hN}$ such that

$$c\|\tilde{u} - u^*\|_1 \leqslant B(\tilde{u} - u^*, w), \quad \|w\|_1 = 1.$$

Hence

$$\begin{aligned} c\|\tilde{u} - u^*\|_1 &\leqslant B(\tilde{u} - u^*, w) = B(\tilde{u} - u, w) + B(u - u^*, w) \\ &\leqslant B(\tilde{u}, w) - \tilde{B}(\tilde{u}, w) + \tilde{B}(\tilde{u}, w) - B(u, w) + c_1\|u - u^*\|_1. \end{aligned}$$

Using (3.1) and (4.11), we see that the middle two terms of this expression combine to vanish. Also, using the properties of K_+,

$$B(\tilde{u}, w) - \tilde{B}(\tilde{u}, w) = \langle K_+\gamma\tilde{u} - K_+\gamma Q_{hN}\tilde{u}, w\rangle \leqslant c_2\|\tilde{u} - Q_{hN}\tilde{u}\|_{H^{1/2}(\Gamma)}.$$

Hence we obtain

$$c\|\tilde{u} - u^*\| \leqslant c_2\|\tilde{u} - Q_{hN}\tilde{u}\|_{H^{1/2}(\Gamma)} + c_1\|u - u^*\|_1.$$

Since $u - \tilde{u} = u - u^* + u^* - \tilde{u}$, we may now use the triangle inequality to obtain the asserted result.

Remark. The last term on the right side of (4.12) is due to the "variational crime" that has been incorporated into the bilinear form $\tilde{B}$. It would be of interest to estimate the size of this term.

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